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Field Equations of Arbitrary Spin in Space-time with Torsion

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A two spinor lagrangian formulation of field equations for massive particle of arbitrary spin is proposed in a curved space-time with torsion. The interaction between fields and torsion is expressed by generalizing the situation of the Dirac equation. The resulting field equations are different (except for the spin-1/2 case) from those obtained by promoting the covariant derivatives of the torsion free equations to include torsion. The non linearity of the equations, that is induced by torsion, can be interpreted as a self-interaction of the particle. The spin-1 and spin-3/2 cases are studied with some details by translating into tensor form. There result the Proca and Rarita-Schwinger field equations with torsion, respectively.

KEY WORDS: field equations in curved space-time; torsion; Proca fields; Rarita-Schwinger fields.

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1. INTRODUCTION

The formulation of arbitrary spin massive field equations in curved space-time has received a consistent form after the papers by Illge (1993); Illge and Schimming (1999); Wünsch (1985); Buchdahl (1982) (see also references therein). It seems natural to extend that formulation to include torsion. This is of interest because torsion adds new degrees of freedom to the theory that could be used to describe further possible physical interactions. Torsion effects have been variously considered in the literature. They generally amount in adding non-linear terms to the field equations (e.g., Hehl *et al.*, 1976; Shapiro, 2002). Some results for specific value of the spin are the following.

The scalar field equation can be generalized to include torsion in different ways. The possible non minimal action extensions are exposed, e.g., in Buchbinder

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et al. (1992). There are scalar field equations with torsion that, even if non linear, still admit plane wave solutions (Zecca, 2002c).

The Dirac equation with torsion can be obtained, on general tensor grounds, from a well known scalar action whose lagrangian is a scalar both under coordinate change and Lorentz rotations (Nakahara, 1990; Weinberg, 1972). The equation can be equivalently obtained by promoting the covariant derivative of the usual spinor formulation to include torsion (Zecca, 2002, 2002b). In Minkowski space-time the Dirac equation with torsion still admits of plane wave solutions (Zecca, 2003). Perturbation of the energy spectrum of the Hydrogen atom induced by torsion results to be (Zecca, 2002a) smaller than the pure gravitational perturbation (Parker, 1980). Effect of torsion on neutrino oscillations has been studied in different space-time models (Cardall and Fuller, 1997; Zhang, 2000, 2005; Alimohammadi and Shariati, 1999; Zecca, 2004). Also effects of torsion on anomalies have been discussed for the Standard Model in curved space-time by Dobado and Maroto (1996).

For what concerns the spin-1 equation with torsion, the formulations by Seitz (1986) and Spinosa (1987) lead to the study of the equation of motion of a "Proca test particle" in the background of a Rieman-Cartan gravity in tensor form. There exist a two spinor formulation of the spin-1 equation with torsion (Zecca, 2002d). It is based on a torsion interaction term that extends the one of the Dirac equation.

The object of the present paper is of generalizing to arbitrary spin value the formulation of the spin-1/2 and spin-1 equations with torsion. To that end the spin-1 case is preliminary reconsidered with improvements and specialization in terms of Proca fields equations with torsion. Then the Lagrangian formulation for arbitrary spin massive field equation with torsion in a general curved space-time is proposed. The field equations are derived by an action principle by varying also with respect to torsion. On account of the assumed special Dirac-like torsion interaction, the resulting equations are different (except for $s = 1/2$) from those obtained by promoting the derivatives of the torsion free case to include torsion. The equations are non linear. They can be interpreted to describe the motion of a particle with self-interaction induced by torsion.

The physical interpretation is detailed in case of spin-3/2. By translating into the tensor language, one recovers the Rarita-Schwinger equations with torsion.

The formalism of the equations becomes rapidly cumbersome by increasing the value of the spin. However, in principle, one could proceed by distinguishing between bosons and fermions, as done here for the spin-1 and spin-3/2 cases. The formalism developed by Illge (1993) for bosons and fermions should give the right indication to proceed in the general case.

Finally note that the present treatment does not cover the most general a priori possible interaction with torsion. Here the interaction with torsion is that of the Dirac equation elementarily extended to higher spin values.

2. ASSUMPTIONS AND PRELIMINARY RESULTS

The following considerations are developed in a four dimensional (curved) space-time of metric tensor g_{ik} with associated spinor and tensor formalism. (For notations and mathematical conventions we refer to Penrose and Rindler (1984) and Chandrasekhar (1983). The standard correspondence between complex tensors of rank n and spinors of type (n, n), that can be realized by the van der Waerden σ-matrices, is denoted by $\leftrightarrow$ (e.g.: $\nabla_\alpha \leftrightarrow \nabla_{AA'} = \sigma^\alpha_{AA'}\nabla_\alpha$). Accordingly any covariant derivative $\widetilde{\nabla}$ can be decomposed into the unique torsion free derivative ∇, that can be derived from the metric tensor through the Chrisoffel coefficients, and a contorsion dependent part

$$\widetilde{\nabla}_\alpha U^\beta = \nabla_\alpha U^\beta + Q_{\alpha\gamma}{}^\beta U^\gamma \tag{1}$$

The contorsion tensor is related to the torsion tensor $\widetilde{T}$ associated to $\widetilde{\nabla}$ by $\widetilde{T}_{\alpha\beta}{}^{\gamma} = -Q_{\alpha\beta}{}^{\gamma} + Q_{\beta\alpha}{}^{\gamma}$. Correspondingly the action of $\widetilde{\nabla}$, ∇ on spinors is characterized by

$$\begin{aligned}
&\widetilde{\nabla}_{AB'}\chi^{PS'} = \nabla_{AB'}\chi^{PS'} + \Theta_{AB'X}{}^{P}\chi^{XS'} + \bar{\Theta}_{A'BX'}{}^{S'}\chi^{PX'}\\
&Q_{abc} \leftrightarrow \Theta_{AX'BC}\,\epsilon_{Y'Z'} + \bar{\Theta}_{X'AY'Z'}\,\epsilon_{BC}\\
&\Theta_{AX'BC} = \Theta_{AX'CB} \qquad (\widetilde{\nabla}_a\epsilon_{AB} = \nabla_a\epsilon_{AB} = 0)
\end{aligned} \tag{2}$$

From the decompositions (1), (2) any spinor or tensor expression can be decomposed into a torsion free and a torsion dependent part. In particular this holds for the scalar curvature $\widetilde{R} = \widetilde{R}_{\alpha\beta}^{\alpha\beta}$. By further symmetrizing the unprimed indexes of Θ

$$\begin{aligned}
&\Theta_{AA'BC} = \Theta_{(A|A'|BC)} - \frac{i}{3}(\epsilon_{AB}Z_{A'C} + \epsilon_{AC}Z_{A'B})\\
&Z_{A'B} = i\Theta_{AA'B}{}^{A} \qquad \left(\bar{Z}_{AB'} = -i\bar{\Theta}_{A'AB'}{}^{A'}\right)
\end{aligned} \tag{3}$$

the Einstein–Hilbert–Cartan Lagrangian density $L_g = \sqrt{|g|}\widetilde{R}$, $(g = \det g_{\mu\nu})$ can be splitted into

$$L_g = \sqrt{|g|}\Bigg\{R - \frac{4}{3}(Z_{B'D}Z^{B'D} + \bar{Z}_{BD'}\bar{Z}^{BD'}) - \Theta_{(A|A'|BC)}\Theta^{(A|A'|BC)} - \bar{\Theta}_{(A'|A|B'C')}\bar{\Theta}^{(A'|A|B'C')}\Bigg\} \tag{4}$$

A divergence term originated from the complete expression of $\widetilde{R}$ has been neglected because no boundary variations will be considered when applying the action priciple. In the following Sections the field equations for massive particles of arbitrary spin s will be derived by starting from a general lagrangian

$L = L_g + L_s$ where L_s expresses the special features of the field under consideration, contains at most first derivatives of the field spinors and an interaction with torsion. The corresponding Euler-Lagrange equations are of the form

$$\frac{\partial L}{\partial \xi^a} - \nabla_{XY'}\left(\frac{\partial L}{\partial \nabla_{XY'} \xi^a}\right) = 0. \tag{5}$$

The equation will be applied to any independent spinor ξ^a on which the Lagrangian depends.

3. SPIN 1 FIELD EQUATION WITH TORSION

The spin-1 equation can be characterized by four spinors $\phi_{AB} = \phi_{BA},\ \chi_{AB'},\ \xi_{A'B'} = \xi_{B'A'},\ \theta_{AB'}$ in such a way that the system of equations satisfied by (ξ, θ) is the complex conjugates of that relative to (ϕ, χ). This can be achieved, by including also torsion, from the following lagrangian:

$$\begin{aligned}
g^{-\frac{1}{2}} L_1 = \ & a\big[\bar{\theta}^{X'(B}\nabla^{A)}_{X'}\phi_{AB} + \chi^{(B|X'|}\nabla^{A)}_{X'}\bar{\xi}_{AB}\big] \\
& + b\big[\phi^{AB}\nabla^{X'}_{(A}\bar{\theta}_{|X'|B)} + \bar{\xi}^{AB}\nabla^{X'}_{(A}\chi_{B)X'}\big] \\
& + \bar{a}\big[\theta^{A(Y'}\nabla^{X')}_{A}\bar{\phi}_{X'Y'} + \bar{\chi}^{(X'|A|}\nabla^{Y')}_{A}\xi_{X'Y'}\big] \\
& + \bar{b}\big[\bar{\phi}^{X'Y'}\nabla^{A}_{(X'}\theta_{|A|Y')} + \xi^{X'Y'}\nabla^{A}_{(X'}\bar{\chi}_{Y')A}\big] \\
& + (a+b)\big[\mu\ \chi_{AX'}\bar{\theta}^{X'A} - \nu\ \phi_{AB}\bar{\xi}^{AB}\big] \\
& + (\bar{a}+\bar{b})\big[\bar{\mu}\ \bar{\chi}_{X'A}\theta^{AX'} - \bar{\nu}\ \bar{\phi}_{X'Y'}\xi^{X'Y'}\big] \\
& + (a+b)\big[A_0\phi_{AB}\big(Z^{X'(A} - \bar{Z}^{(A|X'|}\big)\bar{\theta}^{B)}_{X'} \\
& + C_0\chi_{BX'}\big(Z^{X'(A} - \bar{Z}^{(A|X'|}\big)\bar{\xi}^{B)}_{A}\big)\big] - \\
& - (\bar{a}+\bar{b})\big[B_0\bar{\phi}_{A'B'}\big(\bar{Z}^{X(A'} - Z^{(A'|X|}\big)\theta^{B')}_{X} \\
& + D_0\bar{\chi}_{B'X}\big(\bar{Z}^{X(A'} - Z^{(A'|X|}\big)\xi^{B')}_{A'}\big]\big)
\end{aligned} \tag{6}$$

with $\mu\nu = -m^2$, m the mass of the particles of the fields, $a,\ b$ complex numbers, $a + b \neq 0$. By applying Eq. (5) with $L = L_g + L_1$ and varying with respect to $\bar{\theta},\ \bar{\xi},\ \bar{\chi},\ \bar{\phi},\ \bar{\Theta}_{(A|B'|CD)},\ Z_{B'A},\ \bar{Z}_{AB'}$ one obtains the system of equations

$$\big[\nabla^{A}_{X'} - A_0\big(Z^{A}_{X'} - \bar{Z}^{A}_{X'}\big)\big]\phi_{AB} = -\mu\chi_{BX'} \tag{7}$$

$$\big[\nabla^{X'}_{(A} - C_0\big(Z^{X'}_{(A} - \bar{Z}^{X'}_{(A}\big)\big]\chi_{B)X'} = \nu\phi_{AB} \tag{8}$$

$$\big[\nabla^{Y'}_{A} + D_0\big(Z^{Y'}_{A} - \bar{Z}^{Y'}_{A}\big)\big]\xi_{X'Y'} = -\bar{\mu}\theta_{AX'} \tag{9}$$

$$\left[\nabla^A_{(X'} + B_0\left(Z^A_{(X'} - \bar{Z}^A_{(X'}\right)\right]\theta_{|A|Y')} = \bar{\nu}\xi_{X'Y'} \tag{10}$$

$$Z_{X'A} = \frac{3}{8}\left\{A_0(a+b)\left[\phi_{AB}\bar{\theta}^B_{X'} - \chi_{BX'}\bar{\xi}^B_A\right] - (\bar{a}+\bar{b})\bar{A}_0\left[\bar{\phi}_{X'B'}\theta^{B'}_A - \bar{\chi}_{B'A}\xi^{B'}_{X'}\right]\right\} \tag{11}$$

$$\bar{Z}_{AX'} = -Z_{X'A} \tag{12}$$

$$\Theta_{(A|B'|CD)} = 0 \tag{13}$$

By requiring that the Eqs. (7–10) just obtained coincide with those obtained by varying with respect to $\theta,\ \xi,\ \chi,\ \phi$ respectively and then requiring that the Eqs. (9,10) be the complex conjugates of Eqs. (7,8) one gets

$$B_0 = -\bar{A}_0, \qquad C_0 = -A_0, \qquad D_0 = \bar{A}_0 \tag{14}$$

These conditions has been already used in Eqs. (11,12) and will be assumed to hold in the following.

The spin-1 field equation with torsion are then given by the Eqs. (7–10) where it is understood that Z and $\bar{Z}$ have the explicit expression given by Eqs. (11,12). Therefore the equations come out to be non linear equations. The non linear terms are induced by torsion and can be interpreted as due to self interaction of the particle.

By setting, as in the torsion free case,

$$J^{AX'} = (a+b)A_0\left[\phi^A_B\bar{\theta}^{X'B} + \chi^{X'}_B\bar{\xi}^{AB}\right] - (\bar{a}+\bar{b})\bar{A}_0\left[\bar{\phi}^{X'}_{B'}\theta^{AB'} + \bar{\chi}^A_{B'}\xi^{B'X'}\right] \tag{15}$$

then the spinor $J^{AX'}$ still plays the role of a conserved current. Indeed one can check that $\nabla_{AX'}J^{AX'} = 0$ once the spinors satisfy the Eqs. (7)–(12). Instead $J^{AX'}$ is not conserved with respect to $\widetilde{\nabla}$. One has

$$\widetilde{\nabla}_{AX'}J^{AX'} = -2iZ_{X'A}J^{AX'} \neq 0 \tag{16}$$

[Notice that $J_{X'A}$ and $Z_{X'A}$ are not proportional as erroneously maintained, on account of sign errors, in Zecca (2002d). One can also check that Z is not conserved in the sense of the torsion free derivative ∇].

It is possible to give a physical interpretation of the above scheme, for spin-1 particle, in a special case. Suppose indeed $\theta \equiv \chi$ and choose $\nu = -2,\ \mu = m^2/4$. Define

$$\begin{aligned} U_a &\leftrightarrow \chi_{AX'} \\ H_{ab} &\leftrightarrow \phi_{AB}\,\epsilon_{X'Y'} + \xi_{X'Y'}\,\epsilon_{AB} \end{aligned} \tag{17}$$

Consider then the results

$$\nabla_{[a}U_{b]} \leftrightarrow \frac{1}{2}\left[\epsilon_{X'Y'}\nabla_{(A|Z'|}\chi_{B)}^{Z'} + \epsilon_{AB}\nabla_{C(X'}\chi_{Y')}^{C}\right] \tag{18}$$

$$Q_{cab}U^{c} \leftrightarrow -\frac{i}{3}\left(\epsilon_{X'Y'}Z_{A'(A}\chi_{B)}^{A'} + \epsilon_{AB}Z_{(Y'|X|}\chi_{X')}^{X}\right) \tag{19}$$

$$H_{ab}q^{b} \leftrightarrow -i\left(Z_{X'B}\phi_{A}^{B} + Z_{Y'A}\xi_{X'}^{Y'}\right) \tag{20}$$

$$(q_b \leftrightarrow \Theta_{AY'B}{}^{A} = -iZ_{Y'B})$$

where $a \equiv AX'$, $b \equiv BY'$, $c \equiv CZ'$ and Z has the expression (11). Then by using also the field equations, the Eq. (12) and by choosing $A_0 = i\alpha$ ($\alpha \in \mathbf{R}$), one obtains

$$\nabla_a U_b - \nabla_b U_a = H_{ab} + 3\alpha Q_{cab}U^{c}$$

$$\nabla^{c}H_{ac} = \frac{m^2}{2}U_a - 2\alpha H_{ab}q^{b} \tag{21}$$

The Eq. (21) can be interpreted as Proca fields equations with torsion. Note the in some sense symmetric role torsion plays in the equations. From Eqs. (19, 20, and 11) there also follows that the torsion dependent terms introduces a non linear modification of the original torsion free Proca equations. Finally the conserved current has the Proca fields representation

$$J^{AX'} \leftrightarrow -i\alpha(a+b)\left(H^{a}{}_{b}\bar{U}^{b} - \bar{H}^{a}{}_{b}U^{b}\right) \tag{22}$$

that, of course, is the same of the torsion free scheme (e.g., Illge, 1993).

4. FIELDS EQUATIONS FOR ARBITRARY SPIN WITH TORSION

The scheme for the spin-1 field equation with torsion can be extended to fields of arbitrary spin $s = \frac{n+1}{2}$ by maintaining the Dirac-like interaction with torsion. For spinors ϕ, χ, θ, ξ with symmetry properties $\phi_{AA_1..A_n} = \phi_{(AA_1..A_n)}$, $\xi_{A'A'_1..A'_n} = \xi_{(A'A'_1..A'_n)}$, $\theta_{AX'_1..X'_n} = \theta_{A(X'_1..X'_n)}$, $\chi_{A_1..A_nX'} = \chi_{(A_1A_2..A_n)X'}$ that goal can be reached by the Lagrangian

$$\begin{aligned} g^{-\frac{1}{2}}L_s = {} & a\left[\bar{\theta}^{X'(A_1..A_n}\nabla_{X'}^{A)}\phi_{AA_1..A_n} + \chi_{(A_1..A_n}{}^{X'}\nabla_{A)X'}\bar{\xi}^{AA_1..A_n}\right] \\ & + b\left[\phi_{AB_1..B_n}\nabla^{(A|X'|}\bar{\theta}_{X'}^{B_1..B_n)} + \bar{\xi}^{AB_1..B_n}\nabla_{(A}^{X'}\,\chi_{B_1B_2..B_n)X'}\right] \\ & + \bar{a}\left[\theta^{A(X'_1..X'_n}\nabla_{A}^{X')}\bar{\phi}_{X'X'_1..X'_n} + \bar{\chi}_{(X'_1X'_2..X'_n}{}^{A}\nabla_{|A|Y')}\,\xi^{Y'X'_1X'_2..X'_n}\right] \\ & + \bar{b}\left[\bar{\phi}_{X'X'_1..X'_n}\nabla^{A(X'}\theta_{A}^{X'_1..X'_n)} + \xi^{X'X'_1..X'_n}\nabla_{(X'}^{A}\,\bar{\chi}_{X'_1..X'_n)A}^{A}\right] \\ & + (a+b)\left[\mu\ \chi_{A_1..A_nX'}\bar{\theta}^{X'A_1..A_n} - \nu\ \phi_{AA_1..A_n}\bar{\xi}^{AA_1..A_n}\right] \end{aligned}$$

$$
\begin{aligned}
&+(\bar{a}+\bar{b})\big[\bar{\mu}\ \bar{\chi}_{X'_1..X'_n X}\theta^{XX'_1..X'_n} - \bar{\nu}\ \bar{\phi}_{A'A'_1..A'_n}\xi^{A'A'_1..A'_n}\big]\\
&+(a+b)\Big[A_0\phi_{AB_1..B_n}\Big(Z^{X'(A} - \bar{Z}^{(A|X'|)}\Big)\bar{\theta}_{X'}^{B_1..B_n)}\\
&-A_0\chi_{(B_1..B_n}{}^{X'}\big(Z_{|X'|A)} - \bar{Z}_{A)X'}\big)\bar{\xi}^{AB_1..B_n)}\Big)\Big]\\
&+(\bar{a}+\bar{b})\big[\bar{A}_0\bar{\phi}_{A'B'_1..B'_n}\big(\bar{Z}^{X(A'} - Z^{(A'|X|)}\big)\theta_X^{B'_1..B'_n)}\\
&-\bar{A}_0\bar{\chi}_{(B'_1..B'_n}{}^{X}\big(\bar{Z}_{|X|A')} - Z_{A')X}\Big)\xi^{A'B'_1..B'_n}\big]
\end{aligned}
\tag{23}
$$

Indeed, by varying with respect to $\bar{\theta},\ \bar{\xi},\ \bar{\chi},\ \bar{\phi}$, the Euler-Lagrange Eq. (5) gives now the system of coupled equations

$$\big[\nabla^A_{X'} - A_0\big(Z^A_{X'} - \bar{Z}^A_{X'}\big)\big]\phi_{AA_1A_2..A_n} = -\mu\chi_{A_1A_2..A_nX'} \tag{24}$$

$$\big[\nabla^{X'}_{(A} + A_0\big(Z^{X'}_{(A} - \bar{Z}^{X'}_{(A)}\big)\big]\chi_{B_1..B_n)X'} = \nu\phi_{AB_1..B_n} \tag{25}$$

$$\big[\nabla^{Y'}_{A} + \bar{A}_0\big(Z^{Y'}_{A} - \bar{Z}^{Y'}_{A}\big)\big]\xi_{Y'X'_1..X'_n} = -\bar{\mu}\theta_{AX'_1..X'_n} \tag{26}$$

$$\big[\nabla^{A}_{(X'} - \bar{A}_0\big(Z^{A}_{(X'} - \bar{Z}^{A}_{(X')}\big)\big]\theta_{|A|X'_1..X'_n)} = \bar{\nu}\xi_{X'X'_1..X'_n} \tag{27}$$

$$
\begin{aligned}
Z_{X'A} = {}&\frac{3}{8}A_0(a+b)\big[\phi_{AB_1..B_n}\bar{\theta}_{X'}^{B_1..B_n} - \chi_{B_1..B_nX'}\bar{\xi}_A^{B_1..B_n}\big]\\
&-\frac{3}{8}(\bar{a}+\bar{b})\bar{A}_0\big[\bar{\phi}_{X'B'_1..B'_n}\theta_A^{B'_1..B'_n} - \bar{\chi}_{B'_1..B'_nA}\xi_{X'}^{B'_1..B'_n}\big]
\end{aligned}
\tag{28}
$$

$$\bar{Z}_{AX'} = -Z_{X'A} \tag{29}$$

$$\Theta_{(A|B'|CD)} = 0 \tag{30}$$

The equations that could be obtained by varying $\phi,\ \chi,\ \theta\ \xi$ reproduce the Eqs. (24–30) and, as required, the system (26, 27) is the complex conjugate of the system (24, 25). As a consequence of the results (28, 29), that must be considered into Eqs. (24–27), torsion introduces non linearity in the equations. Also in the present general case the equations are interpreted as to describe a particle with self-interaction produced by torsion. Here, again as in the torsion free case, one can represent the current by the spinor

$$
\begin{aligned}
J^{AX'} = {}&(a+b)A_0\big[\phi^A_{A_1..A_n}\bar{\theta}^{X'A_1..A_n} + \chi_{B_1..B_n}{}^{X'}\bar{\xi}^{AB_1..B_n}\big]\\
&-(\bar{a}+\bar{b})\bar{A}_0\big[\bar{\phi}^{X'}_{A'_1..A'_n}\theta^{AA'_1..A'_n} + \bar{\chi}_{A'_1..A'_n}{}^{A}\xi^{X'A'_1..A'_n}\big]
\end{aligned}
\tag{31}
$$

One can check that, when expressed in terms of the solutions of the equations, the current remains conserved with respect to ∇ but not with respect to $\widetilde{\nabla}$:

$$\nabla_{AX'}J^{AX'} = 0, \qquad \widetilde{\nabla}_{AX'}J^{AX'} = -2iZ_{X'A}J^{AX'} \tag{32}$$

The formalism has been developed by refering, in the torsion free case, to the one given by Illge (1993). One could go on in the analogy by distinguishing between bosonic and fermionic fields. It remains however the problem of a more specific physical interpretation of the equations. For this reason, in the last Section, we study the special case of the spin-3/2 equation with torsion that, as far as the author knows, has not been considered from this point of view.

5. INTERPRETATION OF SPIN-3/2 FIELD EQUATIONS WITH TORSION

The special case $s = 3/2$, $(n = 1)$ of the general scheme of the previous Section can be reported to a tensor form whose physical interpretation can be easily given. To see this, define the field functions (e.g., Illge, 1993)

$$\begin{aligned}\psi_{ab} &\leftrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}\phi_{A_1A_2}{}^{B}\,\epsilon_{X_1'X_2'} + \theta_{X_1'X_2'}{}^{B}\,\epsilon_{A_1A_2}\\ \chi_{A_1A_2X'}\,\epsilon_{X_1'X_2'} + \xi_{X_1'X_2'X'}\,\epsilon_{A_1A_2}\end{pmatrix}\\ \psi^{+}_{ab} &\leftrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}\bar{\xi}_{A_1A_2A}\,\epsilon_{X_1'X_2'} + \bar{\chi}_{X_1'X_2'A}\,\epsilon_{A_1A_2}\\ \bar{\theta}_{A_1A_2}{}^{Y'}\,\epsilon_{X_1'X_2'} + \bar{\phi}_{X_1'X_2'}{}^{Y'}\,\epsilon_{A_1A_2}\end{pmatrix}\end{aligned} \tag{33}$$

where $a \equiv A_1X_1'$, $b \equiv A_2X_2'$ and consider also the operators N, M, $\overrightarrow{\nabla}$, $\overleftarrow{\nabla}$ defined by

$$\begin{aligned}N[\chi]_{A_1A_2A} &\equiv \left(\nabla^{X'}_{(A} + 2A_0Z^{X'}_{(A}\right)\chi_{A_1A_2)X'}\\ N[\xi]_{X_1'X_2'A} &\equiv \left(\nabla^{Y'}_{A} + 2\bar{A}_0Z^{Y'}_{A}\right)\xi_{X_1'X_2'Y'}\\ M[\phi]_{A_1A_2X'} &\equiv \left(\nabla^{A}_{X'} - 2A_0Z^{A}_{X'}\right)\phi_{AA_1A_2}\\ M[\theta]_{X_1'X_2'X'} &\equiv \left(\nabla^{A}_{(X'} - 2\bar{A}_0Z^{A}_{(X'}\right)\theta_{X_1'X_2')A}\end{aligned} \tag{34}$$

$$\begin{aligned}\overrightarrow{\nabla}\,\psi_{ab} &\leftrightarrow i\begin{pmatrix}N[\chi]_{A_1A_2}{}^{A}\,\epsilon_{X_1'X_2'} + N[\xi]_{X_1'X_2'}{}^{A}\,\epsilon_{A_1A_2}\\ -M[\phi]_{A_1A_2Y'}\,\epsilon_{X_1'X_2'} - M[\theta]_{X_1'X_2'Y'}\,\epsilon_{A_1A_2}\end{pmatrix}\\ \left(\psi^{+}\,\overleftarrow{\nabla}\right)_{a'b'} &\leftrightarrow i\begin{pmatrix}-N[\bar{\phi}]_{A_1'A_2'X}\,\epsilon_{X_1X_2} - N[\bar{\theta}]_{XX_1X_2}\,\epsilon_{A_1'A_2'}\\ M[\bar{\chi}]_{A_1'A_2'}{}^{A}\,\epsilon_{X_1X_2} + M[\bar{\xi}]_{X_1X_2}{}^{A}\,\epsilon_{A_1'A_2'}\end{pmatrix}\end{aligned} \tag{35}$$

with $a' \equiv A_1'X_1$, $b' \equiv A_2'X_2$. Then one obtains the following results

$$\begin{aligned}\psi^{+}_{ab}(\overrightarrow{\nabla}\,\psi^{ab}) &= i\sqrt{2}\big[(\bar{\xi}, N[\chi]) + (\bar{\chi}, N[\xi]) + (\bar{\theta}, M[\phi]) + (\bar{\phi}, M[\theta])\big]\\ (\psi^{+}_{ab}\,\overleftarrow{\nabla})\psi^{ab} &= i\sqrt{2}\big[-(N[\bar{\phi}], \theta) - (N[\bar{\theta}], \phi) - (M[\bar{\chi}], \xi) - (M[\bar{\xi}], \chi)\big]\\ \psi^{+}_{ab}\psi^{ab} &= (\bar{\xi}, \phi) + (\bar{\chi}, \theta) - (\bar{\theta}, \chi) - (\bar{\phi}, \xi)\end{aligned} \tag{36}$$

where $(\alpha, \beta) = \alpha_{A_1 A_2 X'} \beta^{A_1 A_2 X'} = -(\beta, \alpha)$ for α, β spinors of the same kind. By taking into account these results and by choosing $\mu = \nu = im\sqrt{2}/4$, $a = b = -i\sqrt{2}$ in Eq. (23), one finds that the Lagrangian $L_{3/2}$ can be written

$$g^{-1/2} L_{3/2} = \psi^{+}_{ab} (\overrightarrow{\nabla}\ \psi^{ab}) - (\psi^{+}_{ab}\ \overleftarrow{\nabla}) \psi^{ab} + m \psi^{+}_{ab} \psi^{ab} \tag{37}$$

Correspondingly the field functions ψ^{ab} are found to satisfy the field equations

$$\overrightarrow{\nabla}\ \psi_{ab} + m\psi_{ab} = 0$$

$$\gamma^a \gamma^b \psi_{ab} = 0 \tag{37}$$

where the γ-matrices are defined through the σ-matrices by $\gamma_a = \sqrt{2} \begin{pmatrix} 0 & \sigma_a^{AX'} \\ \sigma_{aY'B} & 0 \end{pmatrix}$ (e.g., Penrose and Rindler, 1984). The second Eq. (38) is automatically satisfied, as it can be checked, because both ϕ and ξ are assumed to be symmetric spinors in all their indexes. The field functions ψ_{ab} satisfy therefore the Eq. (38) that characterize the Rarita-Schwinger fields (e.g., Illge, 1993; in flat space-time Lurié, 1968). In spite of the strong similarity with the torsion free case, here the equation are non linear. The non linearity of the equations is contained in the action of $\overrightarrow{\nabla}$ because by the defining relations (34), it contains the spinor Z that in turn depends on the unknown spinors as in Eq. (27) for $n = 1$.

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